

THE INCENTRE OF THE CENTRE-CUT PYRAMID: RIGIDITY OF THE TRUNCATED n -CUBIC CONTACT FRAMEWORK, A RADIUS-RATIO IDENTITY, AND THE CaB_6 PACKING MAXIMUM

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ABSTRACT. Cut an n -cube into the $2n$ pyramids that join its centre to its facets and place a ball at the incentre of each pyramid. The centres are the Wythoff points of the $[4, 3^{n-2}, 4]$ kaleidoscope, and in the plane and in space the resulting arrangements (the 4.8.8 tiling, the truncated cubic honeycomb, the boron net of CaB_6) are classical. We study the framework of bars between touching balls. Our main result, proved for every $n \geq 3$ by an explicit block decomposition of the periodic rigidity matrix, is that at a generic wavevector this framework has exactly $n(n-1)/2$ zero modes and $n(n-3)/2$ states of self-stress per cell, with explicit bases, and that with unit springs every shear modulus vanishes while $C_{11}/C_{12} = (n-1)^2$, so that $\nu_{100} = 1/(n^2 - n - 1)$. For $n = 2$ the counts are 2 and 0 and the network is bulk-only. We also record three smaller facts. (i) For a polytope whose centre-to-ridge walls are all mirrors, the hole/ball ratio of the incentre construction equals the classical radius-ratio limit of the polytope formed by the incentres; a second hypothesis (tangential with equal dihedral angles) gives the formula $1/\sin(\varphi/2) - 1$, and we show by example that facet-transitivity and an insphere alone do not suffice. (ii) The maximal packing fraction of hard spheres in the CaB_6 structure type is $(\pi/6)(30\sqrt{2} - 41) \approx 0.74686$, attained only at $r_B/r_{\text{Ca}} = \sqrt{2} - 1$ with Ca held by Ca–Ca contact; it exceeds the fcc value only for $0.4102 < r_B/r_{\text{Ca}} < 0.4159$. (iii) Elementary corollaries of the known hinged-squares mechanism, and a cubic-restricted three-dimensional motion that reaches a six-contact state at 60° but is not jammed. All numbers are reproduced by short scripts.

1. INTRODUCTION

Consider the following planar drawing: draw a square and its diagonals, mark the incentre of each of the four triangles, and join each incentre to the corners of its triangle. The four incentres turn out to be special. A ball of radius equal to the inradius, placed at each incentre, touches its two neighbours across the diagonals, and when the square is repeated over the plane the balls of neighbouring squares touch across the shared side. The same holds in every dimension: if an n -cube of side 2 is cut into the $2n$ pyramids joining its centre to its facets, the incentre of each pyramid lies on an axis at distance $c = 2 - \sqrt{2}$ from the centre, its inradius is $R = \sqrt{2} - 1$, and balls of radius R at these $2n$ points touch along all edges of the cross-polytope they span and across every facet of the cubic tiling (Fig. 1).

What is known. The triangles of the drawing are fundamental domains of the $[4, 4]$ reflection group, the incentre is the point equidistant from the three mirrors, and its orbit is the vertex set of the truncated square tiling 4.8.8. This is the Wythoff construction [3, 2, 12]. In dimension n the incentre is the Wythoff point of the ringing $t_{0,1}\{4, 3^{n-2}, 4\}$, so the balls sit on the vertices of the truncated n -cubic honeycomb with all edges of length $2R$. In three dimensions this vertex set, with equal bonds, is the boron net of CaB_6 in Pauling and Weinbaum’s model [19], catalogued as the RCSR net *cab* [21, 5]. The analogous construction on the rhombic dodecahedron gives the equal-bond UB_{12} boron net, RCSR *ubt*, with $x = 1/6$ [21]. The planar framework of squares joined by rods, which folds the 4.8.8 arrangement into the square tiling, is the “hinged squares with rods” of Flores [7], who credits Stuart [24]; its Poisson ratio -1 along the motion

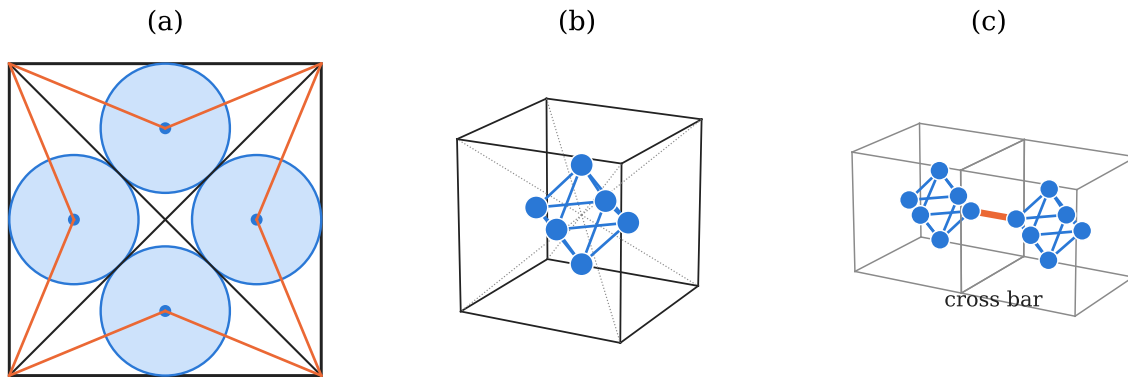


FIGURE 1. (a) The planar construction: incentres of the four triangles cut by the diagonals, with balls of the inradius and lines to the corners. (b) In three dimensions the six incentres of the centre-to-face pyramids form an octahedron of edge $2R$. (c) Two cells of the contact framework: the cross-polytope bars inside each cell and one cross bar between cells.

follows from the symmetry criterion of Mitschke et al. [16]. The three-dimensional framework of octahedra joined by linear bridges is the Prussian-blue topology, whose flat bands of rigid unit modes were found by Goodwin [10]. We claim none of this.

What we add.

- (1) Theorem 1: exact zero-mode and self-stress counts, with explicit bases, for the contact framework of the truncated n -cubic honeycomb in every dimension $n \geq 3$, and its unit-spring elastic tensor, giving $\nu_{100} = 1/(n^2 - n - 1)$. The proof reduces the rigidity matrix to four decoupled scalar systems using the sign-change symmetries of the cross-polytope. The three-dimensional counts are consistent with [10]; we did not find the n -dimensional closed forms in the literature.
- (2) Theorem 2: a short statement and proof that the hole/ball ratio of the incentre construction on a mirror-walled polytope is the classical radius-ratio limit [18] of the polytope of incentres, with the exact conditions under which it holds and fails.
- (3) Theorem 3: the closed form of the maximal packing fraction of the CaB_6 structure type and of the whole curve $\phi_{\max}(\gamma)$. The curve itself was plotted by Horst and Travesset [14].
- (4) Section 6: corollaries of the hinged-squares motion for balls, and a three-dimensional observation, both stated with their limitations.

These results are elementary. Their interest is that they are exact in every dimension and that they connect a single geometric construction to rigidity, packing and crystal chemistry.

2. THE CONSTRUCTION AND THE CONTACT FRAMEWORK

Let $Q_n = [-1, 1]^n$ and let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis. The pyramid over the facet $x_i = 1$ is $\{x \in Q_n : x_i \geq |x_j| \ \forall j\}$; its facets are the base $x_i = 1$ and the walls $x_i = \pm x_j$. A point $t\mathbf{e}_i$ has distance $1 - t$ to the base and $t/\sqrt{2}$ to every wall, so the incentre is $c\mathbf{e}_i$ with

$$c = 2 - \sqrt{2}, \quad R = 1 - c = \sqrt{2} - 1, \quad c\sqrt{2} = 2R.$$

The dots are $p_i^s = s c \mathbf{e}_i$ for $i = 1, \dots, n$ and $s \in \{+1, -1\}$. Two dots p_i^s, p_j^t with $i \neq j$ are at distance $c\sqrt{2} = 2R$; the dot p_i^+ and the dot $p_i^- + 2\mathbf{e}_i$ of the next cell are at distance $2 - 2c = 2R$. All other distances are larger. Tiling by the lattice $2\mathbb{Z}^n$ we obtain:

Definition 1. The *contact framework* $\mathcal{F}_n$ has the dots $p_i^s + 2z$ ($z \in \mathbb{Z}^n$) as joints and bars between touching balls: the $2n(n-1)$ *intra bars* $\{p_i^s, p_j^t\}$, $i < j$, in each cell (the edges of the cross-polytope), and the n *cross bars* $X_i = \{p_i^+, p_i^- + 2\mathbf{e}_i\}$ per cell. Every bar has length $2R$.

Per cell there are $2n$ joints, hence $2n^2$ degrees of freedom, and $E = 2n(n-1) + n = 2n^2 - n$ bars. Each ball touches $2n-1$ others. For the periodic theory we use Bloch displacements $\mathbf{u}(p+2z) = \mathbf{u}(p)e^{2iq \cdot z}$ and write $z_i = e^{2iq_i}$. The rigidity matrix $R(q)$ has one row per bar; the row of a bar from joint a to joint b with unit direction $\hat{d}$ is $\mathbf{u} \mapsto \hat{d} \cdot (\mathbf{u}_b - \mathbf{u}_a)$, where $\mathbf{u}_b$ carries the phase of its cell. A *zero mode* is a vector in $\ker R(q)$ and a *self-stress* a vector in $\ker R(q)^*$. By rank-nullity, $\#\text{zero modes} - \#\text{self-stresses} = 2n^2 - E = n$ at every q , the periodic form of the Maxwell–Calladine count [15, 1, 20]. The proof below uses only the bar directions, so all counting statements hold for dots at $\pm t\mathbf{e}_i$ with any $t \in (0, 1)$.

3. RIGIDITY AND ELASTICITY IN EVERY DIMENSION

Theorem 1. *Let $n \geq 3$.*

- (a) *The cross-polytope framework (intra bars of one cell) is infinitesimally rigid in $\mathbb{R}^n$ and has exactly $n(n-3)/2$ independent self-stresses. A basis: put tension w_{ij} on all four bars $\{p_i^\pm, p_j^\pm\}$, where $w = (w_{ij})_{i < j}$ satisfies $\sum_{j \neq i} w_{ij} = 0$ for every i .*
- (b) *At a wavevector with $q_i \notin \pi\mathbb{Z}$ for all i , $\mathcal{F}_n$ has exactly $n(n-1)/2$ zero modes per cell, namely the infinitesimal rotations $\mathbf{u}(p_i^s) = \Omega p_i^s$ ($\Omega^T = -\Omega$) of each cross-polytope with the same Bloch phase, and exactly $n(n-3)/2$ self-stresses, those of (a) with zero tension on the cross bars. If exactly k of the q_i lie in $\pi\mathbb{Z}$ there are $n(n-1)/2 + k$ zero modes and $n(n-3)/2 + k$ self-stresses. At $q = 0$ there are $n(n+1)/2$ zero modes and $n(n-1)/2$ self-stresses: for every $w \in \mathbb{R}^{\binom{n}{2}}$, tension w_{ij} on the four bars $\{p_i^\pm, p_j^\pm\}$ and tension $\tau_i = \sqrt{2} \sum_{j \neq i} w_{ij}$ on X_i .*
- (c) *Give every bar unit spring constant, so a bar extended by δ stores energy $\delta^2/2$. Impose a homogeneous strain ε and relax all joints. The energy per unit volume is*

$$(1) \quad W(\varepsilon) = \frac{1}{2^n} \left[\frac{2(n-1)^2}{n^2} \sum_i \varepsilon_{ii}^2 + \frac{2}{n^2} \sum_{i \neq j} \varepsilon_{ii} \varepsilon_{jj} \right],$$

independent of the off-diagonal strains. Hence, with $W = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$,

$$C_{11} = \frac{4(n-1)^2}{n^2 2^n}, \quad C_{12} = \frac{4}{n^2 2^n}, \quad C_{44} = 0, \quad \frac{C_{11}}{C_{12}} = (n-1)^2,$$

every component of C involving a shear strain vanishes, and the Poisson ratio for uniaxial stress along a cube axis is

$$\nu_{100} = \frac{C_{12}}{C_{11} + (n-2)C_{12}} = \frac{1}{n^2 - n - 1}.$$

The values of ν_{100} are $1/5, 1/11, 1/19, 1/29, 1/41, \dots$ for $n = 3, 4, 5, 6, 7$ (Fig. 2).

3.1. The block decomposition. Fix one cell. For displacements $\mathbf{u}_i^s = \mathbf{u}(p_i^s)$ write

$$\mathbf{a}_i = \frac{1}{2}(\mathbf{u}_i^+ + \mathbf{u}_i^-), \quad \mathbf{b}_i = \frac{1}{2}(\mathbf{u}_i^+ - \mathbf{u}_i^-),$$

and denote the k -th component of a vector by a superscript. For $i < j$ define the four scalars

$$\alpha_{ij} = a_i^i - a_j^i, \quad \beta_{ij} = a_i^j - a_j^j, \quad \kappa_{ij} = b_i^i + b_j^j, \quad \sigma_{ij} = b_i^j + b_j^i.$$

Lemma 1. *Under a homogeneous strain ε plus displacements $\mathbf{u}$, the first-order extension of the intra bar from p_j^t to p_i^s is*

$$(2) \quad \delta_{ij}^{st} = \frac{1}{\sqrt{2}} \left[c(\varepsilon_{ii} + \varepsilon_{jj}) + \kappa_{ij} + s\alpha_{ij} - t\beta_{ij} - st(2c\varepsilon_{ij} + \sigma_{ij}) \right].$$

For a periodic displacement, the extension of the cross bar X_i is $2R\varepsilon_{ii} - 2b_i^i$. For a Bloch displacement and $\varepsilon = 0$ it is $\mathbf{e}_i \cdot (z_i \mathbf{u}_i^- - \mathbf{u}_i^+)$.

Proof. The bar vector is $\ell = c(\mathbf{s}\mathbf{e}_i - t\mathbf{e}_j)$ with unit vector $\ell/(c\sqrt{2})$, and $\delta = \hat{\ell} \cdot (\varepsilon\ell + \mathbf{u}_i^s - \mathbf{u}_j^t)$. Expand with $\mathbf{u}_i^s = \mathbf{a}_i + s\mathbf{b}_i$ and $s^2 = t^2 = 1$. For X_i the bar vector is $2R\mathbf{e}_i$ and the displacement difference is $\mathbf{u}_i^- - \mathbf{u}_i^+ = -2\mathbf{b}_i$ in the periodic case. $\square$

The four functions $1, s, t, st$ on $\{\pm 1\}^2$ are orthogonal. Hence the four bars between axes i and j impose exactly the four scalar conditions $\kappa_{ij} = \alpha_{ij} = \beta_{ij} = \sigma_{ij} = 0$ when $\varepsilon = 0$, and

$$(3) \quad \frac{1}{2} \sum_{s,t} (\delta_{ij}^{st})^2 = (c(\varepsilon_{ii} + \varepsilon_{jj}) + \kappa_{ij})^2 + \alpha_{ij}^2 + \beta_{ij}^2 + (2c\varepsilon_{ij} + \sigma_{ij})^2.$$

The rigidity problem has split into four independent linear systems: one in the n diagonal entries b_i^i (via κ), one in the n^2 entries of the $\mathbf{a}_i$ (via α, β), and one in the $n(n-1)$ off-diagonal entries b_i^j (via σ).

3.2. Proof of Theorem 1(a). *The a-block.* The conditions $\alpha_{ij} = \beta_{ij} = 0$ for all $i < j$ say that, for each k , all the numbers a_j^k ($j = 1, \dots, n$) equal a_k^k . So the solutions are $\mathbf{a}_j = \mathbf{v}$ for all j , an n -dimensional space of translations. The map $\mathbf{a} \mapsto (\alpha, \beta)$ goes from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n(n-1)}$ with an n -dimensional kernel, so it is onto: these $n(n-1)$ conditions are independent.

The σ -block. Each σ_{ij} involves the two variables b_i^j, b_j^i , which occur in no other condition. So the $\binom{n}{2}$ conditions are independent, and the solutions are the matrices $B = (b_i^j)$ with zero symmetric off-diagonal part.

The κ -block. The conditions $b_i^i + b_j^j = 0$ for all $i < j$ form the system $x_i + x_j = 0$ over the edges of the complete graph K_n . For $n \geq 3$ any triangle i, j, k gives $x_i = -x_j = x_k = -x_i$, so $x = 0$. The coefficient matrix is the unsigned incidence matrix of K_n , of rank n , and its left kernel, the vectors w on edges with $\sum_{j \neq i} w_{ij} = 0$ at each vertex, has dimension $\binom{n}{2} - n = n(n-3)/2$.

Together, the zero modes are $\mathbf{a}_j = \mathbf{v}$, $b_i^i = 0$ and B antisymmetric, that is $\mathbf{u}_i^s = \mathbf{v} + s\Omega'\mathbf{e}_i$ with Ω' antisymmetric; with $\Omega = \Omega'/c$ these are the translations plus the rotations $\mathbf{u}_i^s = \mathbf{v} + \Omega p_i^s$. So the kernel has dimension $n + n(n-1)/2$ and the framework is infinitesimally rigid. The rank is $2n^2 - n(n+1)/2 = n(3n-1)/2$, and the number of self-stresses is $2n(n-1) - n(3n-1)/2 = n(n-3)/2$.

For the explicit basis, expand a stress on the four bars of the pair (i, j) in the characters $1, s, t, st$. By (2), its pairing with a displacement is a combination of $\kappa_{ij}, \alpha_{ij}, \beta_{ij}, \sigma_{ij}$ with the character coefficients. Since the $\mathbf{a}$ - and σ -blocks have no relations, the s, t and st coefficients must vanish, and the character-1 coefficients w_{ij} must satisfy $\sum_{i < j} w_{ij} \kappa_{ij} = \sum_i b_i^i \sum_{j \neq i} w_{ij} \equiv 0$, i.e. $\sum_{j \neq i} w_{ij} = 0$. Physically: equal tension on the four bars of the square spanned by $p_i^\pm, p_j^\pm$ pulls p_i^s inward along $\mathbf{e}_i$ with force $\sqrt{2}w_{ij}$, and these pulls cancel at each joint. $\square$

3.3. Proof of Theorem 1(b). Intra bars join joints of the same cell, so their rows carry no phase. A zero mode of $\mathcal{F}_n$ therefore restricts on each cell to an infinitesimal motion of the cross-polytope, which by (a) is $\mathbf{u}_i^s = \mathbf{v} + \Omega p_i^s$ with complex $\mathbf{v}, \Omega$. Since $\mathbf{e}_i \cdot \Omega \mathbf{e}_i = 0$, both $\mathbf{e}_i \cdot \mathbf{u}_i^+$ and $\mathbf{e}_i \cdot \mathbf{u}_i^-$ equal v^i , and by Lemma 1 the cross bar X_i requires

$$(z_i - 1)v^i = 0.$$

If $z_i \neq 1$ for all i then $\mathbf{v} = 0$ and Ω is free: $n(n-1)/2$ zero modes. If exactly k of the z_i equal 1, the corresponding v^i are free: $n(n-1)/2 + k$ zero modes. The self-stress counts follow from rank-nullity, $\#\text{self-stresses} = \#\text{zero modes} - n$. The stresses of (a) remain stresses of $\mathcal{F}_n$ with zero cross-bar tension, because intra rows carry no phase; at generic q they are all of them. At $q = 0$, take any w and the tensions of the statement. At p_i^s the intra tensions give the force $-\sqrt{2}s\mathbf{e}_i \sum_j w_{ij}$ and the cross bar gives $s\tau_i\mathbf{e}_i$, so the choice $\tau_i = \sqrt{2}\sum_j w_{ij}$ balances every joint. These stresses form a space of dimension $\binom{n}{2} = n(n-1)/2$, which is the full count. $\square$

3.4. Proof of Theorem 1(c). A homogeneous macroscopic strain produces the same bar extensions in every cell. In a Bloch decomposition of the relaxation over any periodic supercell, only the $q = 0$ component couples linearly to ε ; the other components add a non-negative energy and are optimally zero. So it suffices to relax over one-cell-periodic displacements, for which Lemma 1 and (3) give the energy per cell

$$\mathcal{E} = \sum_{i < j} \left[(c(\varepsilon_{ii} + \varepsilon_{jj}) + \kappa_{ij})^2 + \alpha_{ij}^2 + \beta_{ij}^2 + (2c\varepsilon_{ij} + \sigma_{ij})^2 \right] + 2 \sum_i (R\varepsilon_{ii} - b_i^i)^2.$$

The variables $\mathbf{a}$ appear only in α, β ; choose $\mathbf{a} = 0$. The off-diagonal b_i^j appear only in σ_{ij} , and each σ_{ij} can be chosen freely, so choose $\sigma_{ij} = -2c\varepsilon_{ij}$. This removes every off-diagonal strain from the energy: all shear moduli vanish. (The relaxed motion keeps each cross-polytope undeformed; the shear is taken up by first-order rotations of the cross bars.)

With $e_i = \varepsilon_{ii}$, $x_i = b_i^i$ and $y_i = x_i + ce_i$, and using $c + R = 1$, the remaining energy is

$$\mathcal{E} = \sum_{i < j} (y_i + y_j)^2 + 2 \sum_i (e_i - y_i)^2.$$

Its Hessian in y is $2(nI + J)$, with J the all-ones matrix, which is positive definite. The stationarity conditions $ny_k + \sum_j y_j = 2e_k$ give $\sum_j y_j = \frac{1}{n} \sum_j e_j$ and

$$y_k = \frac{1}{n} \left(2e_k - \frac{1}{n} \sum_j e_j \right).$$

For a quadratic $y^T My - 2(2e) \cdot y + 2|e|^2$ the minimum is $2|e|^2 - 2e \cdot y$, which evaluates to

$$\mathcal{E}_{\min} = \frac{2(n-2)}{n} \sum_i e_i^2 + \frac{2}{n^2} \left(\sum_i e_i \right)^2 = \frac{2(n-1)^2}{n^2} \sum_i e_i^2 + \frac{2}{n^2} \sum_{i \neq j} e_i e_j.$$

Dividing by the cell volume 2^n gives (1). For uniaxial stress along $\mathbf{e}_1$ the conditions $\partial W / \partial e_j = 0$ ($j \geq 2$) read $C_{12}e_1 + (C_{11} + (n-2)C_{12})e_\perp = 0$, which gives the stated ν_{100} , and $C_{11} + (n-2)C_{12} = (4/n^2 2^n)((n-1)^2 + n-2)$ yields $1/(n^2 - n - 1)$. $\square$

Remark 1 (The case $n = 2$). The proof of (c) does not use $n \geq 3$, so (1) holds for $n = 2$: $C_{11} = C_{12} = 1/4$, all shear moduli vanish, and only the bulk modulus is non-zero; formally $\nu_{100} = 1$. The counts in (a) and (b) change, because the κ -block for K_2 is the single condition $b_1^1 + b_2^2 = 0$: the four-bar square has one internal flex. At generic q this flex couples to the translations through the cross bars and survives, giving 2 zero modes and 0 self-stresses per cell; at $q = 0$ there are 3 zero modes and 1 self-stress. These agree with the numerical check.

Remark 2. (1) All statements hold for dots at $\pm t\mathbf{e}_i$ with any $0 < t < 1$: the counts use only directions, and in (c) the cross bars have length $2 - 2t$, so the identity $t + (1 - t) = 1$ replaces $c + R = 1$. The incentre is the unique t at which all bars have the same length, so that “unit springs” means “identical bars”. (2) For $n = 3$ the framework is the Prussian-blue topology, and the flat bands of (b) are the rigid unit modes of [10]; our contribution is the closed form in n and the explicit bases. (3) At generic q the self-stresses live inside the cross-polytopes, so for

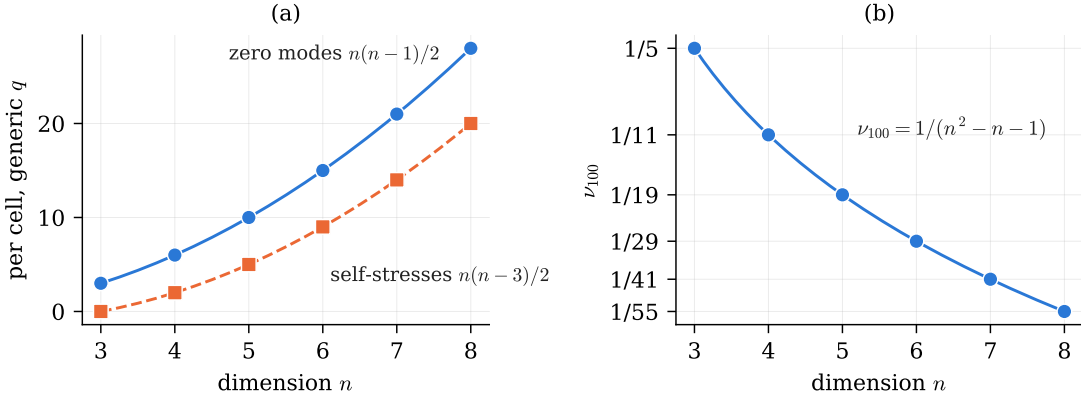


FIGURE 2. (a) Zero modes and self-stresses per cell at a generic wavevector (lines: Theorem 1; markers: rank of the numerically assembled rigidity matrix). (b) Poisson ratio ν_{100} with unit springs (line: formula; markers: full numerical relaxation).

$n \geq 4$ the framework is over-braced at the scale of one cell and under-braced at the scale of the lattice at the same time.

Numerical check. The script `proofs/rigidity_nd.py` builds $R(q)$ and the relaxed elastic tensor directly from coordinates, without using the decomposition, for $n = 2, \dots, 8$. It confirms every count in Theorem 1 at random q , at $q = 0$ and on the planes $q_1 = \dots = q_k = 0$; that the stated bases span the self-stress spaces and the rotations span the generic zero modes; and the elastic constants to 10^{-12} with full, unconstrained relaxation, for $c = 2 - \sqrt{2}$ and for two other values of c . The script `proofs/proof_steps.py` checks (2), (3) and the minimisation symbolically for $n = 2, \dots, 8$.

4. A RADIUS-RATIO IDENTITY

Let $P \subset \mathbb{R}^n$ be a convex polytope with the origin O in its interior. For a facet F let $\Pi_F = \text{conv}(O \cup F)$, and for a ridge $E = F \cap G$ let W_E be the hyperplane through O and E (the *wall*). Suppose each Π_F has an *inscribed ball*, a ball in Π_F tangent to every facet of Π_F ; its centre d_F is the *dot* and its radius r_F .

Lemma 2. *If a convex polytope has a ball tangent to all its facet hyperplanes from inside, that ball is its unique largest inscribed ball.*

Proof. Let the facets be $n_k \cdot x \leq h_k$ with unit n_k , the tangent ball $B(x, r)$, so $n_k \cdot x + r = h_k$. If $B(y, r')$ lies inside with $r' \geq r$, then $n_k \cdot (y - x) \leq r - r' \leq 0$ for all k . Since the polytope is bounded there are $\lambda_k > 0$ with $\sum \lambda_k n_k = 0$, which forces $n_k \cdot (y - x) = 0$ for all k and $r' = r$; as the n_k span $\mathbb{R}^n$, $y = x$. $\square$

Theorem 2. *Assume one of the following.*

- (H1) *Every wall W_E is a mirror of P , and Π_F has an inscribed ball for one (hence every) facet.*
- (H2) *P has an insphere centred at O , and the angle φ between the normals of any two adjacent facets is the same for all ridges.*

Then all dots have the same radius r and the same distance D from O ; the balls of two facets sharing a ridge touch, at a point of the common wall; no two balls overlap; and the ball of radius

$D - r$ centred at O touches every ball. Consequently the hole/ball ratio is

$$\frac{D - r}{r} = \frac{2D}{e} - 1, \quad e = 2r,$$

which is the classical radius-ratio limit of the polytope Q of dots: anions of radius $e/2$ on a sphere of radius D , touching their neighbours and a central cation. Under (H1) P is facet-transitive and Q is vertex-transitive. Under (H2) Q is a scaled polar dual of P , inscribed with all edges equal to $2r$, and

$$\frac{D - r}{r} = \frac{1}{\sin(\varphi/2)} - 1;$$

conversely every convex inscribed polytope with equal edges and its circumcentre inside arises in this way.

Proof. (H1). Let $E = F \cap G$ and let ρ_E be the reflection in W_E . It is a symmetry of P fixing O and E pointwise, so it permutes the two facets through E . It cannot fix F : the plane of F contains $E \subset W_E$ and does not contain O , so if it were ρ_E -invariant it would be perpendicular to W_E and F would lie on both sides of E in its plane. So $\rho_E(F) = G$, $\rho_E(\Pi_F) = \Pi_G$, and an inscribed ball of Π_F maps to one of Π_G ; by Lemma 2 $\rho_E(B_F) = B_G$. The ball B_F is tangent to W_E at a point fixed by ρ_E , so B_F and B_G touch there, and $r_F = r_G$, $|Od_F| = |Od_G|$. The ridge graph of a polytope is connected, so r and D are constant, and the reflections generate a group acting transitively on facets. Balls lie in pyramids with disjoint interiors, so they do not overlap. The remaining claims are immediate.

(H2). Let ϱ be the inradius, $\mathbf{n}_F$ the unit outer normal and $T_F = \varrho \mathbf{n}_F \in F$ the tangency point. Since O is equidistant from the planes of F and G and lies on the inner side of both, W_E is the internal bisector of the dihedral angle at E , with normal proportional to $\mathbf{n}_F - \mathbf{n}_G$. The angle θ between $\mathbf{n}_F$ and W_E satisfies $\sin \theta = \mathbf{n}_F \cdot (\mathbf{n}_F - \mathbf{n}_G) / |\mathbf{n}_F - \mathbf{n}_G| = \sin(\varphi/2)$. The point $d = (\varrho - r)\mathbf{n}_F$ on the segment OT_F therefore has distance r from the base and $(\varrho - r)\sin(\varphi/2)$ from each wall of Π_F ; with $r = \varrho \sin(\varphi/2) / (1 + \sin(\varphi/2))$ these agree, so $d = d_F$ and $D = \varrho - r = r / \sin(\varphi/2)$. The reflection in W_E exchanges $\mathbf{n}_F$ and $\mathbf{n}_G$, hence d_F and d_G , whose distance is $2D \sin(\varphi/2) = 2r$. The dots $D\mathbf{n}_F$ are $D\varrho$ times the vertices $\mathbf{n}_F/\varrho$ of the polar dual, whose edges correspond to ridges of P . Conversely, if Q is inscribed in a sphere about O with equal edges, its polar has all facets at the same distance from O , and the angle between the normals at the ridge dual to an edge qq' is $\angle qOq'$, the same for all edges. $\square$

Remark 3 (Scope). (1) (H1) holds for the regular polytopes in every dimension, the rhombic dodecahedron, the k -gonal bipyramids of *every* height (for which Q is always the uniform k -prism: the mirrors force all edges equal), and the Catalan duals of Wythoffian uniform polyhedra, whose walls are the perpendicular bisector planes of the dual edges. (2) (H2) adds the duals of snub polyhedra and the tetragonal trapezohedron of the single shape dual to the uniform square antiprism. (3) Facet-transitivity and an insphere do not suffice: for tetragonal trapezohedra polar to square antiprisms of height 0.7 and 1.3 times the uniform one, every pyramid has an inscribed ball, but the distances between adjacent dots range over $[1, 1.017] \cdot 2r$ and $[1, 1.006] \cdot 2r$, so half the contacts open (`proofs/radius_ratio.py`). (4) A prism P is not facet-transitive: for the triangular prism with square sides the pyramids have different inradii and the theorem does not apply.

Unreachable shapes. The CN7 and CN9 polyhedra of the radius-ratio tables, and the dodecahedral CN8, are not vertex-transitive, so no polytope satisfying (H1) produces them. Under (H2) Q must be inscribed with all edges equal. For the triangulated shapes (pentagonal bipyramid CN7, snub disphenoid CN8, tricapped trigonal prism CN9) all faces would then be equilateral,

TABLE 1. Hole/ball ratio of the incentre construction (LP for the inscribed balls) against the classical radius-ratio limit of Q . All rows satisfy (H1) except the last two, which satisfy (H2).

P	Q (coordination)	hole/ball	classical
tetrahedron	tetrahedron (CN4)	0.2247	$\sqrt{3/2} - 1 = 0.2247$
cube	octahedron (CN6)	0.4142	$\sqrt{2} - 1 = 0.4142$
octahedron	cube (CN8)	0.7321	$\sqrt{3} - 1 = 0.7321$
dodecahedron	icosahedron (CN12)	0.9021	0.9021
rhombic dodecahedron	cuboctahedron (CN12)	1.0000	1
triangular bipyramid, any height	trigonal prism (CN6)	0.5275	0.5275
icosahedron	dodecahedron (CN20)	1.8025	1.8025
tesseract	16-cell (4D)	0.4142	0.4142
pentagonal icositetrahedron	snub cube	1.6874	1.6874
trapezohedron (one shape)	square antiprism (CN8)	0.6453	0.6453

and by the classical enumeration of the eight convex deltahedra the only ones with these combinatorics are the Johnson solids J_{13} , J_{84} , J_{51} . None is inscribed: for unit edge their vertex orbits lie at distances 0.851 and 0.526, 0.930 and 0.677, 0.764 and 0.996 from the centre. So these limits are unreachable by either route. We did not examine the capped CN7 shapes under (H2). The square antiprism is reached only under (H2) and only at one shape; under (H1) every adjacent pair of dots is exchanged by a reflection symmetry, and no reflection symmetry of the antiprism exchanges the ends of a lateral edge. The construction thus explains one structural limitation of the radius-ratio rule (it applies naturally only to mirror-generated, equal-edged polyhedra); it does not address the softness and size effects behind most of the rule's failures [8].

5. THE CaB_6 PACKING MAXIMUM

Tiled by the cubic lattice, the dots of the cube are the boron positions of Pauling and Weinbaum's equal-bond model of CaB_6 [19]: Ca at $1a$ (0, 0, 0), B at $6f$ ($x, \frac{1}{2}, \frac{1}{2}$) and permutations, $x = (\sqrt{2} - 1)/2$, at which the B–B distances inside and between octahedra are equal [21]. Consider hard spheres of radius r_L on the Ca sites and r_S on the B sites, with x and the lattice parameter a free, and $\gamma = r_S/r_L$.

Theorem 3. *The maximal packing fraction of the CaB_6 structure type at size ratio γ is*

$$\phi_{\max}(\gamma) = \begin{cases} \frac{\pi}{6}(1 + 6\gamma^3), & 0 < \gamma \leq \sqrt{2} - 1, \\ \frac{\pi}{6}(\sqrt{2} - 1)^3(6 + \gamma^{-3}), & \gamma \geq \sqrt{2} - 1. \end{cases}$$

Its global maximum is $\phi^* = \frac{\pi}{6}(30\sqrt{2} - 41) = 0.746865$, attained only at $\gamma = \sqrt{2} - 1$, $x = (\sqrt{2} - 1)/2$, $a = 2r_L$. There the Ca spheres touch each other and each B sphere touches five B spheres, but Ca and B do not touch: $d_{\text{Ca-B}} = 0.73681a > r_L + r_S = 0.70711a$. Moreover $\phi_{\max}(\gamma) > \pi/(3\sqrt{2})$ (fcc) exactly when

$$\left(\frac{\sqrt{2}-1}{6}\right)^{1/3} = 0.41023 < \gamma < (4 + 7\sqrt{2})^{-1/3} = 0.41591.$$

Proof. Set $a = 1$; then $\phi = \frac{4\pi}{3}(r_L^3 + 6r_S^3)$. The constraints include Ca–Ca: $2r_L \leq 1$; B–B inside an octahedron: $2r_S \leq \sqrt{2}(\frac{1}{2} - x)$; B–B between octahedra: $2r_S \leq 2x$. The last two give

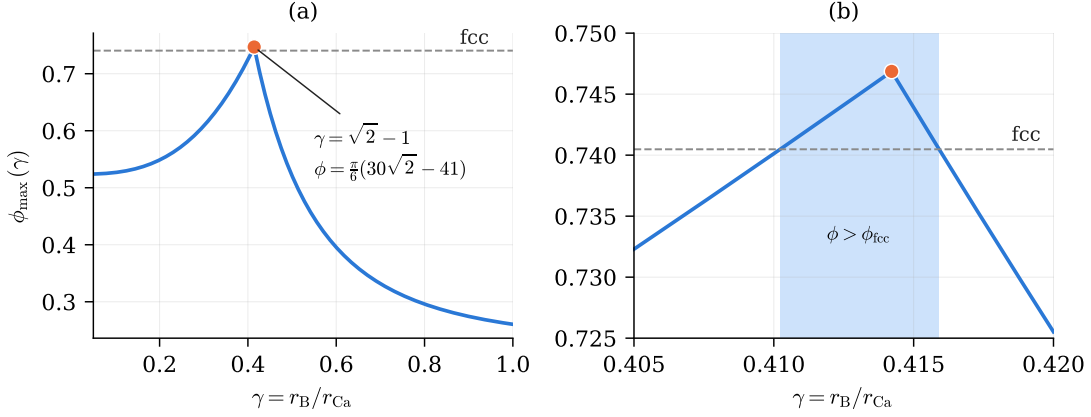


FIGURE 3. (a) Maximal packing fraction of the CaB_6 type versus size ratio (Theorem 3); dashed line: fcc. (b) Detail of the cusp; shaded: the window where it exceeds fcc.

$r_S \leq m(x) = \min((\frac{1}{2} - x)/\sqrt{2}, x) \leq m^* = (\sqrt{2} - 1)/2$, with equality only at $x^* = (\sqrt{2} - 1)/2$. Hence $\phi \leq \frac{4\pi}{3} \cdot \frac{1}{8}(1 + 6\gamma^3)$ from $r_L \leq \frac{1}{2}$, and $\phi = \frac{4\pi}{3}r_S^3(\gamma^{-3} + 6) \leq \frac{4\pi}{3}m^{*3}(6 + \gamma^{-3})$ from $r_S \leq m^*$. The first bound is smaller for $\gamma \leq \sqrt{2} - 1$ and the second for $\gamma \geq \sqrt{2} - 1$. Both are attained at $x = x^*$: for $\gamma \leq \sqrt{2} - 1$ take $r_L = \frac{1}{2}$, $r_S = \gamma/2 \leq m^*$; for $\gamma \geq \sqrt{2} - 1$ take $r_S = m^*$, $r_L = m^*/\gamma \leq \frac{1}{2}$. The Ca–B distance is $\sqrt{x^{*2} + \frac{1}{2}} = 0.73681$, and $r_L + r_S \leq 0.73681$ holds in the first case for $\gamma \leq 0.4736$ and in the second for $\gamma \geq 0.3910$, so it holds throughout. All other pair distances at x^* exceed these (checked over a 5^3 block). The maximum of $\phi_{\max}$ is at the junction $\gamma = \sqrt{2} - 1$, where $1 + 6(\sqrt{2} - 1)^3 = 30\sqrt{2} - 41$. The window follows from $\frac{\pi}{6}(1 + 6\gamma^3) > \frac{\pi}{3\sqrt{2}} \iff 6\gamma^3 > \sqrt{2} - 1$ and $(\sqrt{2} - 1)^3(6 + \gamma^{-3}) > \sqrt{2} \iff \gamma^{-3} > \sqrt{2}(5\sqrt{2} + 7) - 6 = 4 + 7\sqrt{2}$. $\square$

Remark 4. (1) The curve was plotted, for this and 23 other lattices, by Horst and Travesset [14, Fig. 3]; the peak position can be read from their figure. We did not find the closed forms in text. (2) The maximum is a cusp (Fig. 3), and the window above fcc is narrow and asymmetric, -0.0040 and $+0.0017$ about $\sqrt{2} - 1$. (3) CaB_6 is not a densest binary packing at this size ratio: the NaCl structure at $\gamma = \sqrt{2} - 1$ has $\phi = \pi(5 - 3\sqrt{2})/3 = 0.7931$, and CaB_6 is absent from the densest binary packings of Hopkins et al. [13]. The Murray–Sanders list of structures beating fcc [17] does not mention CaB_6 , consistent with the window being so narrow. CaB_6 -type nanoparticle superlattices are observed [23].

6. REMARKS ON FINITE MOTIONS

In this section the balls are joined by centre-to-centre bars of length $2R$ (balls may roll on each other); a model with pins at the contact points behaves differently.

The plane. The 4.8.8 framework of squares and rods folds into the square tiling [7, 24], a relative of the rotating-squares mechanism [11]: alternate squares rotate by $\pm\theta$, the cross bars stay on the axes, and the period becomes $L(\theta) = 2R + 2c \cos \theta$ in both directions, so the motion is isotropic with $\nu = -1$ along it, as the criterion of [16] predicts. The motion is the symmetric path; the periodic framework has other infinitesimal flexes, including a shear.

Corollary 1. *Along this motion the closest pair of non-bonded balls has gap $2\sqrt{2}c \cos(\theta + \pi/4)$ (numerically no other pair comes closer), so the first new contact occurs at $\theta = \pi/4$. There*

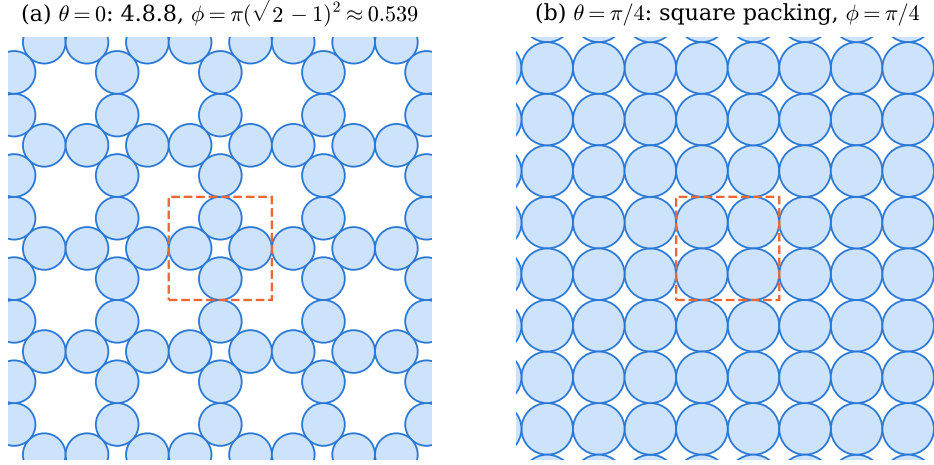


FIGURE 4. Endpoints of the hinged-squares motion of [7, 24], drawn with balls of radius R and bars of length $2R$. Dashed: one cell.

$L = 4R$, the linear strain is $L/2 - 1 = -R^2 = -(3 - 2\sqrt{2})$, the dots form a square lattice of spacing $2R$, and the packing fraction has risen from $\pi(\sqrt{2} - 1)^2 = 0.5390$ to $\pi/4$ (Fig. 4). Among dots placed on the axis mirror lines at $\pm t\mathbf{e}_i$, only $t = 2 - \sqrt{2}$ ends in a square lattice with a single contact distance (for $t < 2 - \sqrt{2}$ this part is numerical).

The last statement is clear at $\theta = \pi/4$: the dot spacings along each axis then alternate between $t\sqrt{2}$ and $2 - 2t$, equal only at the incentre. For $t < 2 - \sqrt{2}$ the motion continues past $\pi/4$ and stops at a state with three contacts per ball (numerical scan, `proofs/fold.py`). The range $[0.539, \pi/4]$ is the flexibility window of this framework in the sense of [22]; the kagome analogue, triangles twisted by 30° into the triangular packing, is of the same type [6, 25]. We expect these facts to be regarded as folklore.

Space. Rotate the octahedra of the three-dimensional framework alternately by $\pm\theta$ about [111] (the analogue of the $a^-a^-a^-$ tilt [9]) and keep the lattice cubic. The bars then fix the lattice parameter $a(\theta) = K_1 + \sqrt{4R^2 - 2K_2^2}$, with $K_2 = \frac{2}{3}c(1 - \cos\theta)$ and $K_1 = 2c\cos\theta + K_2$. At $\theta = 60^\circ$ we find $a = 8c/3$ exactly, every ball has exactly six contacts, and the packing fraction is $27\pi\sqrt{2}/256 = 0.4686$ (from $\pi R^3 = 0.2233$ at $\theta = 0$); numerically no earlier contact occurs. This is *not* a jammed state. If the lattice may become trigonal, with lattice vectors the rows of $\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$, all bar lengths are still preserved, and the set of non-overlapping states connected to $\theta = 0$ on a $1^\circ \times 0.01$ grid contains states of packing fraction up to 0.648 (near $\theta = 31^\circ$, with large shear). The six-contact net at 60° has coordination sequence 6, 17, 42, 68, 110, 153, 212, 272, 348, 425, which matches RCSR `sms` (sphere-packing type 6/3/h2) term by term [21]. We checked the coordination sequence only, not an isomorphism, and record the match as an observation.

7. OUTLOOK

The rigidity counts of Theorem 1 rest on the sign-change symmetries of the cross-polytope, and the same character decomposition should apply to other frameworks built from centrally symmetric clusters joined along symmetry axes. A full description of the configuration space of these ball-and-bar frameworks, beyond the symmetric paths of Section 6, is open, as is whether the incentre is always the unique dot position whose motions end in single-edge-length packings. Finally, whether people perceive the centre of a triangle nearer its incentre or its centroid appears

not to have been tested; an early study of perceived centres of polygons found them closest to the area barycentre among the mathematical centres it compared [4].

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CODE AVAILABILITY

Every number in this paper is produced by the scripts in `proofs/` and `figures/`; the accompanying `README.md` lists which script gives which result.

REFERENCES

1. C. R. Calladine, *Buckminster Fuller's "tensegrity" structures and Clerk Maxwell's rules for the construction of stiff frames*, International Journal of Solids and Structures **14** (1978), no. 2, 161–172.
2. J. H. Conway, H. Burgiel, and C. Goodman-Strauss, *The symmetries of things*, A K Peters, Wellesley, MA, 2008.
3. H. S. M. Coxeter, *Regular polytopes*, 3rd ed., Dover, New York, 1973.
4. M. Davi, N. Yakimoff, N. Bocheva, and V. Kefaloff, *The relative roles of vertices and sides in determining perceptual centres within shapes*, Acta Neurobiologiae Experimentalis **53** (1993), no. 2, 367–375.
5. O. Delgado Friedrichs, M. O’Keeffe, and O. M. Yaghi, *Three-periodic nets and tilings: semiregular nets*, Acta Crystallographica A **59** (2003), no. 6, 515–525.
6. A. Donev, S. Torquato, F. H. Stillinger, and R. Connelly, *Jamming in hard sphere and disk packings*, Journal of Applied Physics **95** (2004), no. 3, 989–999.
7. Alfinio Flores, *Hinged tilings*, North American GeoGebra Journal **6** (2017), no. 1, Section 2.9 and Fig. 9, “Hinged squares with rods”.
8. J. George, D. Waroquiers, D. Di Stefano, G. Petretto, G.-M. Rignanese, and G. Hautier, *The limited predictive power of the Pauling rules*, Angewandte Chemie International Edition **59** (2020), no. 19, 7569–7575.
9. A. M. Glazer, *The classification of tilted octahedra in perovskites*, Acta Crystallographica B **28** (1972), 3384–3392.
10. A. L. Goodwin, *Rigid unit modes and intrinsic flexibility in linearly bridged framework structures*, Physical Review B **74** (2006), 134302.
11. J. N. Grima and K. E. Evans, *Auxetic behavior from rotating squares*, Journal of Materials Science Letters **19** (2000), no. 17, 1563–1565.
12. B. Grünbaum and G. C. Shephard, *Tilings and patterns*, W. H. Freeman, New York, 1987.
13. A. B. Hopkins, F. H. Stillinger, and S. Torquato, *Densest binary sphere packings*, Physical Review E **85** (2012), 021130.
14. N. Horst and A. Travesset, *Prediction of binary nanoparticle superlattices from soft potentials*, Journal of Chemical Physics **144** (2016), 014502, Fig. 3: packing fraction versus size ratio for 24 lattices including CaB_6 .
15. J. C. Maxwell, *On the calculation of the equilibrium and stiffness of frames*, Philosophical Magazine, Series 4 **27** (1864), no. 182, 294–299.
16. H. Mitschke, G. E. Schröder-Turk, K. Mecke, P. W. Fowler, and S. D. Guest, *Symmetry detection of auxetic behaviour in 2D frameworks*, EPL **102** (2013), 66005.
17. M. J. Murray and J. V. Sanders, *Close-packed structures of spheres of two different sizes II. The packing densities of likely arrangements*, Philosophical Magazine A **42** (1980), no. 6, 721–740.
18. L. Pauling, *The principles determining the structure of complex ionic crystals*, Journal of the American Chemical Society **51** (1929), no. 4, 1010–1026.
19. L. Pauling and S. Weinbaum, *The structure of calcium boride, CaB_6* , Zeitschrift für Kristallographie **87** (1934), 181–182.
20. S. C. Power, *Polynomials for crystal frameworks and the rigid unit mode spectrum*, Philosophical Transactions of the Royal Society A **372** (2014), 20120030.
21. Reticular Chemistry Structure Resource, *Nets cab, ubt and sms*, <http://rcsr.net> (raw data file 3dall.txt), Accessed 2026-10-12.

- 22. A. Sartbaeva, S. A. Wells, M. M. J. Treacy, and M. F. Thorpe, *The flexibility window in zeolites*, Nature Materials **5** (2006), no. 12, 962–965.
- 23. E. V. Shevchenko, D. V. Talapin, N. A. Kotov, S. O’Brien, and C. B. Murray, *Structural diversity in binary nanoparticle superlattices*, Nature **439** (2006), no. 7072, 55–59.
- 24. D. Stuart, *Polyhedral and mosaic transformations*, Student Publications of the School of Design, North Carolina State University **12** (1963), no. 1, 2–38, Page range as cited by Flores.
- 25. K. Sun, A. Souslov, X. Mao, and T. C. Lubensky, *Surface phonons, elastic response, and conformal invariance in twisted kagome lattices*, Proceedings of the National Academy of Sciences USA **109** (2012), no. 31, 12369–12374.

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